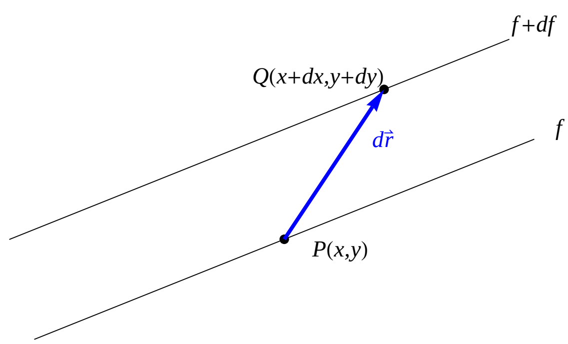
Components of the gradient vector

- start with function $f : \mathbb{R}^2 \to \mathbb{R}$ and point P in the domain where, in a zoomed-in view, the level curve through P and nearby level curves are parallel lines
- define *gradient vector* $\vec{\nabla} f$ as vector that
  - points in direction of greatest rate of change (so perpendicular to level curve through P)
  - has magnitude $\|\vec{\nabla} f\|$ equal to that greatest rate of change
- introduce cartesian coordinates to have $P(x, y)$
- consider infinitesimal displacement $d\vec{r} = dx\,\hat{\imath} + dy\,\hat{\jmath}$ consisting of displacements dx and dy in the x and y directions, respectively
- for the displacement $d\vec{r}$, there is a corresponding infinitesimal change df in the function values



- relate df to dx and dy using partial derivatives as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

- note that each term on the right side is a contribution to the change df that has the form

(rate of change in f with respect to change in coordinate)$\times$(size of change in coordinate)

- factor this using the dot product as

$$df = \left(\frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath} \right) \cdot (dx\,\hat{\imath} + dy\,\hat{\jmath})$$

- in this product of two vectors, the first vector has information about rate of change and the second vector has information about displacement

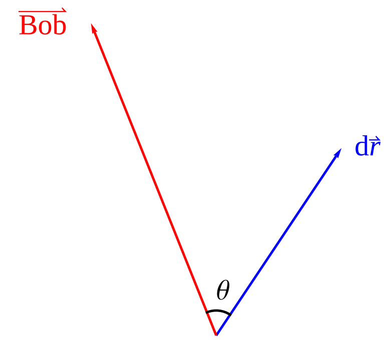
- for convenience, name the first vector in the product $\overrightarrow{\text{Bob}}$ so have

$$\overrightarrow{\text{Bob}} = \frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath}$$

  and can write

$$df = \overrightarrow{\text{Bob}} \cdot d\vec{r} \qquad (1)$$

- now argue that $\overrightarrow{\text{Bob}}$ is equal to the gradient vector $\vec{\nabla} f$



- start by writing the geometric expression for the dot product in (1) to get

$$df = \|\overrightarrow{\text{Bob}}\| \, \|d\vec{r}\| \, \cos\theta = \|\overrightarrow{\text{Bob}}\| \, ds \, \cos\theta$$

  where θ is the angle between $\overrightarrow{\text{Bob}}$ and $d\vec{r}$ and $ds = \|d\vec{r}\|$

- the rate of change in f for a displacement $d\vec{r}$ is the ratio of df to ds
- dividing through by ds in the previous relation gives

$$\text{rate of change in } f \text{ for displacement } d\vec{r}\text{:} \quad \frac{df}{ds} = \|\overrightarrow{\text{Bob}}\| \, \cos\theta \qquad (2)$$

- now consider all displacements $d\vec{r}$ having the same magnitude $ds = \|d\vec{r}\|$ while allowing the direction to vary so the only variable in (2) is θ
- since $\cos\theta$ has values between -1 and 1, the greatest rate of change is for $\cos\theta = 1$ corresponding to $\theta = 0$
- so, the greatest rate of change is for a displacement in the direction of $\overrightarrow{\text{Bob}}$ with magnitude

$$\left.\frac{df}{ds}\right|_{\theta=0} = \|\overrightarrow{\text{Bob}}\| \, \cos 0 = \|\overrightarrow{\text{Bob}}\|(1) = \|\overrightarrow{\text{Bob}}\|$$

- in other words, $\overrightarrow{\text{Bob}}$ is a vector that
  - points in direction of greatest rate of change
  - has magnitude equal to that greatest rate of change
- thus, $\overrightarrow{\text{Bob}}$ is equal to the gradient vector $\vec{\nabla} f$
- recalling the definition of $\overrightarrow{\text{Bob}}$, we have

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\,\hat{\imath} + \frac{\partial f}{\partial y}\,\hat{\jmath}$$

- this result gives us a way to compute the components of a gradient vector $\vec{\nabla} f$ if we have a formula for f in terms of cartesian coordinates
- knowing that $\overrightarrow{\text{Bob}} = \vec{\nabla} f$, can relate df to $\vec{\nabla} f$ by rewriting (1) as

$$df = \vec{\nabla} f \cdot d\vec{r}$$